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Analytical and numerical studies on the influence of multiplication operators for the ill-posedness of inverse problems ^{*†}

MELINA FREITAG [‡] and BERND HOFMANN [§]

Abstract

In this paper we deal with the degree of ill-posedness of linear operator equations $Ax = y$, $x \in X$, $y \in Y$, in the Hilbert space $X = Y = L^2(0, 1)$, where $A = M \circ J$ is a compact operator that may be decomposed into the simple integration operator J with a well-known decay rate of singular values and a multiplication operator M determined by the multiplier function m . This case occurs for example for nonlinear operator equations $F(x) = y$ with a forward operator $F = N \circ J$ where N is a Nemytskii operator. Then the local degree of ill-posedness of the nonlinear equation at a point x_0 of the domain of F is investigated via the Fréchet derivative of F which has the form $F'(x_0) = M \circ J$. We show the restricted influence of such multiplication operators M mapping in $L^2(0, 1)$.

If the multiplier function m has got zeros, the determination of the degree of ill-posedness is not trivial. We are going to investigate this situation and provide analytical tools as well as their limitations. For power and exponential type multiplier functions with essential zeros we will show by using several numerical approaches that the unbounded inverse of the injective multiplication operator does not influence the local degree of ill-posedness. We provide a conjecture, verified by several numerical studies, how these multiplication operators influence the singular values of $A = M \circ J$.

Finally we analyze the influence of those multiplication operators M on the possibilities of Tikhonov regularization and corresponding convergence rates. We investigate the role of approximate source conditions in the method of Tikhonov regularization for linear and nonlinear ill-posed operator equations. Based on the studies on approximate source conditions we indicate that only integrals of m and not the decay of multiplier functions near zero determines the convergence behavior of the regularized solution.

MSC2000 subject classification: 65R30, 47A52 47B06, 65J20, 45C05, 47H30, 47B33, 47J06

Keywords: Inverse problems, linear and nonlinear ill-posed problems, singular values, degree of ill-posedness, multiplication operator, Nemytskii operator, Tikhonov regularization, Fréchet derivative, convergence rates, source conditions

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1 Introduction

In this paper, we deal with the potential for ill-posedness of specific multiplication operators in $L^2(0, 1)$. In particular we focus on the situation when the associated multiplier (weight) function occurring in the forward operator of a linear inverse problem or in the Fréchet derivative of the forward operator in a nonlinear inverse problem has essential zeros of power type or exponential type.

Let X and Y be infinite dimensional Hilbert spaces over the field of real numbers and let $\|\cdot\|$ denote the generic norm in both spaces. On the one hand we consider inverse problems that can be written as linear operator equations

$$Ax = y \quad (x \in X, y \in Y), \quad (1)$$

where the injective and bounded linear forward operators $A \in \mathcal{L}(X, Y)$ are assumed to have a non-closed range, i.e. $R(A) \neq \overline{R(A)}$. This implies an unbounded inverse $A^{-1} : R(A) \rightarrow X$ and leads to ill-posed equations (1).

In particular we consider the class of compact composite linear integral operators $A = M \circ J$, defined as

$$[Ax](s) = m(s) \int_0^s x(t) dt \quad (0 \leq s \leq 1), \quad (2)$$

for $X = Y = L^2(0, 1)$. Since A from (2) is assumed to be compact, we always have $R(A) \neq \overline{R(A)}$ and an ill-posed linear operator equation (1) for that A . By definition the operator A from (2) is a composition of the injective simple linear integration operator J defined as

$$[Jx](s) = \int_0^s x(t) dt \quad (0 \leq s \leq 1) \quad (3)$$

and the multiplication operator M defined as

$$[Mx](t) = m(t)x(t) \quad (0 \leq t \leq 1).$$

We only focus on multiplier functions m satisfying

$$m \in L^1(0, 1), \quad |m(t)| > 0 \text{ a.e. on } [0, 1], \quad (4)$$

and impose additional conditions on m such that $A = M \circ J$ is a compact operator in $L^2(0, 1)$. Note that (4) implies the injectivity of the operators M and A .

It is well-known that J is a compact linear operator in $L^2(0, 1)$. Moreover, M is a bounded linear operator and hence $A = M \circ J$ a compact linear operator in $L^2(0, 1)$ if $m \in L^\infty(0, 1)$. For the multiplier functions m we preferably focus on two families both satisfying the condition (4). First we deal with the family of power type functions

$$m(t) = t^\alpha \quad (0 \leq t \leq 1, \quad \alpha > -1) \quad (5)$$

which have a zero at $t = 0$ for $\alpha > 0$ and belong to $L^\infty(0, 1)$ for $\alpha \geq 0$. Consequently, for $\alpha \geq 0$ the composite operator A is compact. On the other hand, for $-1 < \alpha < 0$ we have $m \in L^1(0, 1)$ and a weak pole at $t = 0$, but nevertheless A is compact in $L^2(0, 1)$ (see [40]). As a second family we consider the exponential type functions

$$m(t) = \exp\left(-\frac{1}{t^\alpha}\right) \quad (0 < t \leq 1) \quad (6)$$

with exponent $\alpha > 0$ which can be extended continuously to $[0, 1]$ by setting $m(0) = 0$. All such functions m belong to $L^\infty(0, 1)$ and therefore A is also a compact operator in $L^2(0, 1)$.

Furthermore we consider nonlinear ill-posed problems (see [8, Chapter 10] and [36]) written as an operator equation

$$F(x) = y \quad (x \in D(F) \subseteq X, \ y \in Y), \quad (7)$$

where the nonlinear forward operator $F : D(F) \subseteq X \rightarrow Y$ with closed, convex domain $D(F)$ is assumed to be continuous. If F is smoothing enough, in particular if F is compact and weakly closed, then local ill-posedness of equations (7) at the solution point $x_0 \in D(F)$ in the sense of [23, Definition 2] (see also [34, Definition 7.1.1]) arises, i.e., the solutions x do not stably depend on the data y in a neighborhood of x_0 .

We especially consider the class of nonlinear equations (7) with composite nonlinear operators

$$F = N \circ J : D(F) \subset L^2(0, 1) \rightarrow L^2(0, 1)$$

in $X = Y = L^2(0, 1)$, defined as

$$[F(x)](s) = \kappa(s, [Jx](s)) \quad (0 \leq s \leq 1; \ x \in D(F)) \quad (8)$$

with J from (3) and domains

$$D(F) = \{x \in L^2(0, 1) : x(t) \geq c_0 \geq 0 \text{ a.e. on } [0, 1]\}. \quad (9)$$

Here, N defined as

$$[N(z)](t) = \kappa(t, z(t)) \quad (0 \leq t \leq 1; \ z \in D(N)) \quad (10)$$

is a nonlinear Nemytskii operator (see, e.g., [2]) generated by a function $k(t, v)$ ($(t, v) \in [0, 1] \times [0, \infty)$). If the generator function κ is sufficiently smooth, then the operator

$$N : D(N) \subset L^2(0, 1) \rightarrow L^2(0, 1)$$

defined by formula (10) with $D(N) = \{z \in L^2(0, 1) : z(t) \geq 0 \text{ a.e. on } [0, 1]\}$ is continuous. Moreover, as a consequence of the compactness of J and the continuity of N the composition $F = N \circ J$ is compact, continuous, weakly continuous and hence a weakly closed nonlinear operator. In view of the compactness and weak closedness of F the equation (7) is locally ill-posed on the whole domain (9) (cf. the arguments in the context of [17, Corollary 5.2]). Under stronger assumptions on the smoothness of κ the operator F from (8) is even Fréchet differentiable with a Fréchet derivative

$$F'(x_0) = A = M \circ J \quad (11)$$

of the form (2) at the point x_0 where the corresponding multiplier function m depends on the point $x_0 \in D(F)$ and attains the form

$$m(t) = \kappa_v(t, [Jx_0](t)) \quad (0 \leq t \leq 1).$$

Here, κ_v is the partial derivative of the generator function $\kappa(t, v)$ with respect to the second variable v .

Multiplication operators as in (2) arise in natural sciences and technology as well as in finance. Examples of nonlinear inverse problems (7) with nonlinear operators of the form (8) and Fréchet derivatives (11) mapping in $L^2(0, 1)$ are presented in [21] (see also [19] and [20, pp.57 and pp.123]). For more details on an example in finance with multiplication operators of exponential type we refer to [17].

We are both interested in the degree of ill-posedness of composite operators like (2) as well as in the influence of multiplication operators on Tikhonov regularization. We will present both analytical and numerical results in order to show that the influence of the multiplication operator on the degree of ill-posedness of (2) is limited.

2 The degree of ill-posedness for composite integral operators

We are looking for the asymptotics of the positive singular values $\sigma_n(A)$ ($n = 1, 2, \dots$) arranged in a decreasing order of the compact composite operator $A = M \circ J$ from (2) mapping in $L^2(0, 1)$ for different multiplier functions m (see also [11] and [25]). This decay rate of singular values to zero as $n \rightarrow \infty$ determines the degree of ill-posedness of equation (1) which can be expressed by a finite real number $\mu = \mu(A)$ if $\sigma_n(A) \asymp n^{-\mu}$ (see, e.g., [8, p.40], [18, p.31] and [27, p.235]).¹ Otherwise, we can also use an interval of ill-posedness

$$[\underline{\mu}(A), \overline{\mu}(A)] = \left[\liminf_{n \rightarrow \infty} \frac{-\log \sigma_n(A)}{\log n}, \limsup_{n \rightarrow \infty} \frac{-\log \sigma_n(A)}{\log n} \right]$$

as introduced in [24], where the values 0 and ∞ may occur as left and right ends of the interval.

For compact linear Fredholm integral equations of the first kind (1) in $X = Y = L^2(0, 1)$ with operators

$$[Ax](s) = \int_0^1 k(s, t)x(t)dt, \quad (0 \leq s \leq 1) \quad (12)$$

that are defined by square-integrable kernel functions the degree of ill-posedness of (1) grows with an increasing kernel smoothness. Namely, Fredholm integral operators (12) with

$$\int_0^1 \int_0^1 (k(s, t))^2 ds dt < \infty$$

are compact and of Hilbert-Schmidt type. Then we have a well-known connection between the kernel smoothness and the decay rate of the singular values, which is due to Chang (see [39] or [6] and [22]) and follows from a classical theorem by Hille and Tamarkin on eigenvalues of integral operators, stated in [1]. In particular, for any Hilbert-Schmidt operator A

$$\sigma_n(A) = o(n^{-\frac{1}{2}}) \quad \text{as } n \rightarrow \infty$$

holds. Unfortunately, in most cases only a lower bound for the left end $\underline{\mu}(A)$ of the ill-posedness interval of (1) can be found from the knowledge of the smoothness properties of the kernel function. In the case of the composite integral operator A from (2), which is of Volterra type, the kernel attains the form

$$k(s, t) = \begin{cases} m(s) & (0 \leq t \leq s \leq 1) \\ 0 & (0 \leq s \leq t \leq 1). \end{cases} \quad (13)$$

This kernel is square-integrable for $m \in L^2(0, 1)$ and has got a finite jump on the diagonal $s = t$. We get $\underline{\mu}(A) \geq \frac{1}{2}$, i.e., the operator equation is then at least ill-posed of degree $\frac{1}{2}$.

Another approach to evaluate the decay rate of singular values is the minimum-maximum principle by Poincaré and Fischer (see, e.g., [5, Lemma 4.18] or [18, Lemma 2.44]) with the following lemma as a consequence:

Lemma 2.1 (Spectral equivalence) *Let $A : X \rightarrow Y$ and $B : X \rightarrow Z$ be compact linear operators mapping between Hilbert spaces X , Y and Z such that*

$$c\|Bx\| \leq \|Ax\| \leq C\|Bx\| \quad \forall x \in X$$

¹Here the notation $a_n \asymp b_n$ for sequences of positive numbers a_n and b_n denotes the existence of positive constants c_1 and c_2 such that $c_1 \leq a_n/b_n \leq c_2$ for all sufficiently large n . If moreover $\lim_{n \rightarrow \infty} a_n/b_n = 1$ we write $a_n \sim b_n$.

with some constants $0 < c \leq C < \infty$. Then the ordered singular values of A and B satisfy

$$c \sigma_n(B) \leq \sigma_n(A) \leq C \sigma_n(B), \quad n = 1, 2, \dots$$

Hence the intervals of ill-posedness of both operators coincide.

If we set $X = Y = Z = L^2(0, 1)$ this lemma applies to the composite operator A from (2) with $B = J$ provided that

$$0 < c \leq |m(t)| \leq C < \infty \quad \text{a.e. on } [0, 1].$$

Then A and J are spectrally equivalent and because of the well-known fact

$$\sigma_n(J) = \frac{1}{\pi \left(n - \frac{1}{2}\right)} \asymp \frac{1}{n}$$

the ill-posedness interval of A degenerates to a singleton and (1) is only mildly ill-posed with ill-posedness degree

$$\mu(A) = \mu(J) = 1. \quad (14)$$

Note that from Lemma 2.1 we cannot conclude this for multiplier functions m with zeros as for example for functions from the families (5) and (6) of power and exponential type.

Another way to estimate the ill-posedness of the composite operator $A = M \circ J$ is to consider

$$\begin{aligned} [A^* A x](\tau) &= \int_0^1 k(s, \tau) \left(\int_0^1 k(s, t) x(t) dt \right) ds, \\ &= \int_0^1 \left(\int_0^1 k(s, \tau) k(s, t) ds \right) x(t) dt \quad (0 \leq \tau \leq 1). \end{aligned}$$

We easily find the kernel $K(t, \tau)$ of the integral operator $A^* A$ as a function of the kernel of A itself in the form

$$K(t, \tau) = \int_0^1 k(s, \tau) k(s, t) ds. \quad (15)$$

This function is Lipschitz continuous, but not continuously differentiable on the unit square. Using the definition of the kernel function (13) we get for $m \in L^2(0, 1)$

$$K(t, \tau) = \int_{\max(t, \tau)}^1 m^2(s) ds. \quad (16)$$

By using the results of Reade (see [32] and [33]) on the eigenvalues λ_n of operators with Lipschitz continuous, self-adjoint, positive definite kernels $K(t, \tau)$ we obtain

$$\lambda_n(A^* A) = \mathcal{O}(n^{-2})$$

implying

$$\sigma_n(A) = \mathcal{O}(n^{-1})$$

and $\underline{\mu}(A) \geq 1$ for the singular values in the case of square-integrable multiplier functions with zeros. Note that continuously differentiable functions $K(t, \tau)$ would lead to a singular value asymptotics $\mathcal{O}(n^{-1})$ that would contradict (14), but such kernels do not occur for the composite operator $A = M \circ J$.

Results on the singular value asymptotics of (2) with weak poles in the multiplier function m have been found by Vu Kim Tuan and Gorenflo [38] in the more general context of fractional

integral operators J_ν of order $\nu > 0$ in $L^2(0, 1)$, where also the weighted version $A_\nu = M \circ J_\nu$ of the form

$$[A_\nu x](s) = s^\alpha \int_0^s \frac{(s-t)^{\nu-1}}{\Gamma(\nu)} x(t) dt \quad (0 < s \leq 1) \quad (17)$$

has been considered. They proved the asymptotics

$$\sigma_n(A_\nu) \asymp \sigma_n(J_\nu) \asymp n^{-\nu} \quad \text{for all } \frac{-\nu}{2} < \alpha < 0$$

and all $\nu > 0$. Consequently, the weak pole function m does not change the singular value asymptotics of J_ν here. Recently, in the paper [25] written of the second author joint with von Wolfersdorf for the specific situation $\nu = 1$ of our operator (2) this result was extended to general power type functions (5) as follows:

Proposition 2.2 *For the ordered singular values of the compact linear operator A_1 in $L^2(0, 1)$ defined by*

$$[A_1 x](s) = s^\alpha \int_0^s x(t) dt \quad (0 < s \leq 1)$$

with exponent $\alpha > -1$ we have the asymptotics

$$\sigma_n(A_1) \sim \frac{1}{(\alpha+1)\pi n} = \left(\int_0^1 m(t) dt \right) \frac{1}{\pi n}.$$

As a consequence we also have

$$\sigma_n(A) \asymp n^{-1}$$

for every compact composite linear operator A from (2) with a multiplier function m satisfying inequalities

$$c t^{\alpha_2} \leq |m(t)| \leq C t^{\alpha_1} \quad \text{a.e. on } [0, 1]$$

with some constants $-1 < \alpha_1 \leq \alpha_2$ and $c, C > 0$.

For the proof of this proposition we refer to the original paper [25]. These results for special multiplier functions and the numerical results in the subsequent section motivate the following conjecture:

Conjecture 2.3 *We conjecture the asymptotic behaviour*

$$\sigma_n(A) \sim \left(\int_0^1 m(t) dt \right) \sigma_n(J) \quad (18)$$

for A from (2) whenever the multiplier function m satisfies the inequalities

$$0 < m(t) \leq C t^\alpha \quad \text{a.e. on } [0, 1]. \quad (19)$$

for some $\alpha > -1$.

This conjecture is based on three comprehensive numerical case studies which we will going to examine in the next section. In particular, we compare the behavior of power functions (5) and exponential functions (6) and point out that the decay of singular values $\sigma_n(A)$ is uniformly proportional to $1/n$ in all three studies, where as in formula (18) the integral $\int_0^1 m(t) dt$ occurred as essential factor in all studies. Note that the condition (19) immediately implies $m \in L^1(0, 1)$ or more detailed $0 < \int_0^1 m(t) < \infty$. A stringent proof of formula (18), however, seems to be missing up to now even for the family (6).

3 Numerical approaches

3.1 A finite difference method for the Sturm-Liouville problem

For evaluating the singular values $\sigma > 0$ of the composite operator A from (2) we first search for the eigenvalues $\lambda = \frac{1}{\sigma^2} > 0$ that occur in the Sturm-Liouville problem to the ordinary differential equation

$$-(a(\tau)u'(\tau))' = \lambda u(\tau) \quad \text{with} \quad a(\tau) = \frac{1}{m^2(\tau)} \quad (0 < \tau < 1) \quad (20)$$

and boundary conditions

$$u(1) = 0 \quad \text{and} \quad \lim_{\tau \rightarrow 0} a(\tau)u'(\tau) = 0.$$

For details on the transformation of the integral operator (2) into the Sturm-Liouville problem we refer to [11] and [25]. For a numerical approach we assume that a is continuously differentiable and u is twice continuously differentiable on the open interval $(0, 1)$.

By using equidistant meshes with meshsize $h = 1/N$ and applying central differences to (20) we get a tridiagonal matrix whose eigenvalues have to be determined. The boundary values are discretized by

$$u(1) \approx u_N = 0,$$

on the right-hand side and by a forward difference for the derivative on the left-hand side boundary

$$\lim_{\tau \rightarrow 0} a(\tau)u'(\tau) \approx \frac{a_1 u_1 - a_0 u_0}{h} = 0.$$

Since $a_0 = a(0)$ is not defined we introduce a variable ϵ , which has to be close to zero and we define

$$a_0 := a(\epsilon).$$

Then, we can vary ϵ in order to take it as close to zero as possible and get the following algorithm:

Algorithm 3.1 (Finite difference method)

1. Determine the matrix $\underline{A}_h \in \mathbb{R}^{N+1, N+1}$ given by

$$\underline{A}_h = \begin{bmatrix} -\frac{a_0}{h} & \frac{a_1}{h} & 0 & \dots & \dots \\ \frac{a_2 - a_0}{4h^2} - \frac{a_1}{h^2} & \frac{2a_1}{h^2} & \frac{a_0 - a_2}{4h^2} - \frac{a_1}{h^2} & 0 & \dots \\ 0 & \frac{a_3 - a_1}{4h^2} - \frac{a_2}{h^2} & \frac{2a_2}{h^2} & \frac{a_1 - a_3}{4h^2} - \frac{a_2}{h^2} & \dots \\ \ddots & \ddots & \ddots & \ddots & \dots \\ \dots & 0 & \frac{a_N - a_{N-2}}{4h^2} - \frac{a_{N-1}}{h^2} & \frac{2a_{N-1}}{h^2} & \frac{a_{N-2} - a_N}{4h^2} - \frac{a_{N-1}}{h^2} \\ \dots & \dots & \dots & \dots & 1 \end{bmatrix}.$$

2. Compute the algebraic eigenvalues $\lambda_1^{approx} \leq \lambda_2^{approx} \leq \dots \leq \lambda_{N+1}^{approx}$ of the matrix \underline{A}_h .

The scheme is consistent and stable (see [11] for further details) and therefore convergent. We apply this scheme to multiplier functions of the families (5) and (6). In Figure 1 and Figure 2 we find logarithmic scale plots of the eigenvalues approximated by the finite difference method for power functions $m(s) = s^\alpha$, i.e. $a(s) = \frac{1}{s^{2\alpha}}$, and exponential functions $m(s) = e^{-\frac{1}{s^\alpha}}$, i.e. $a(s) = e^{\frac{2}{s^\alpha}}$, respectively, for different exponents α . The plots suggest that

$$\lambda_n^{approx} \approx f(\alpha) n^2$$

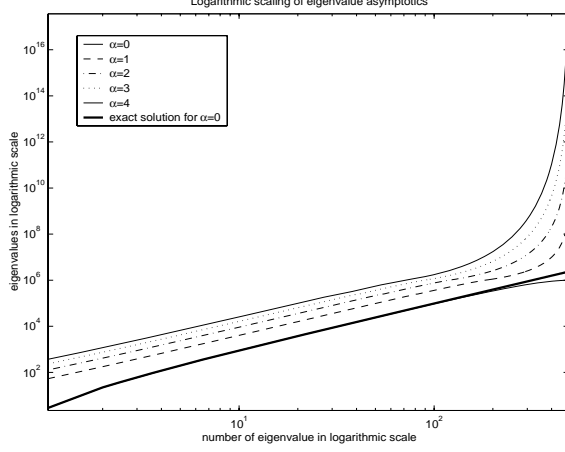


Figure 1: Approximated eigenvalues of Sturm-Liouville problem $-(au')' = \lambda u$ for $N = 500$, multiplier function $m(s) = s^\alpha$ and different values for α and exact eigenvalues for $\alpha = 0$ in logarithmic scales

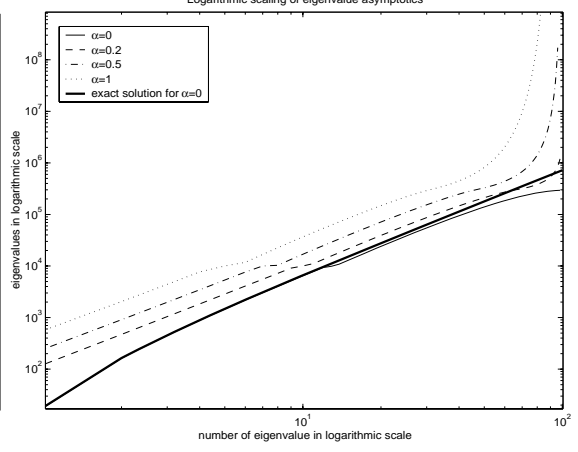


Figure 2: Approximated eigenvalues of Sturm-Liouville problem $-(au')' = \lambda u$ for $N = 100$, multiplier function $m(s) = e^{-\frac{1}{s^\alpha}}$ and different values for α and exact eigenvalues for $\alpha = 0$ in logarithmic scales

for both power and exponential type functions and some monotonically increasing functions f depending on the exponents α . By several comparisons of the coefficients for different values of α and using the fact that $\lambda_n = \frac{1}{\sigma_n^2(J)} = \frac{\pi^2(2n-1)^2}{4}$ holds for $m(s) \equiv 1$, where we can determine the eigenvalues of the Sturm-Liouville operator analytically, we found that the approximated singular values of the integral operator A under consideration with multiplier function (5) have the asymptotics

$$\sigma_n^{approx} = \frac{1}{\sqrt{\lambda_n^{approx}}} \sim \frac{1}{(\alpha + 1)\pi n}. \quad (21)$$

This result corresponds to Proposition 2.2. Unfortunately a similar result to (21) cannot be found for exponential type multiplier functions (6). For those we may only conjecture (18) which we will verify later.

In [11] also non-equidistant meshes were considered. It turns out that the condition number of the matrix \underline{A}_h becomes very large (both for equidistant and non-equidistant meshes) due to the nontrivial boundary condition at the left hand side boundary. Therefore we now leave the Sturm-Liouville problem and apply numerical schemes directly to the integral equation

$$[Ax](s) = \int_0^1 k(s, t) x(t) dt = y(s) \quad (0 \leq s \leq 1) \quad (22)$$

with kernel k from (13) that avoid the complicated boundary conditions occurring in the Sturm-Liouville approach.

3.2 A Galerkin method for the original integral equation

In this subsection we assume that $m \in L^2(0, 1)$. Hence, the operator A from (22) is a compact Hilbert-Schmidt operator acting in $L^2(0, 1)$ with the singular system $\{\sigma_i, u_i, v_i\}$ such that a singular value expansion (SVE)

$$k(s, t) = \sum_{j=1}^{\infty} \sigma_j u_j(t) v_j(s) \quad (0 \leq s \leq 1, 0 \leq t \leq 1)$$

holds with kernel function $k(s, t)$ from (13), where we have the Hilbert-Schmidt norm of A as

$$\|A\|_{HS}^2 = \int_0^1 \int_0^1 (k(s, t))^2 ds dt = \sum_{j=1}^{\infty} \sigma_j^2.$$

Let $\underline{A}_N \in \mathbb{R}^{N \times N}$ denote a real square matrix of dimension N that is obtained from the operator A by discretization. Then the algebraic singular value decomposition (SVD) of this matrix is given by (see, e.g., [12])

$$\underline{A}_N = \underline{U}_N \underline{\Sigma}_N \underline{V}_N^T = \sum_{j=1}^N s_j^{(N)} \mathbf{u}_j^{(N)} \left(\mathbf{v}_j^{(N)} \right)^T,$$

where

$$\begin{aligned} \underline{\Sigma}_N &= \text{diag}(s_1^{(N)}, s_2^{(N)}, \dots, s_N^{(N)}) \in \mathbb{R}^{N \times N}, \\ \underline{U}_N &= [\mathbf{u}_1^{(N)}, \mathbf{u}_2^{(N)}, \dots, \mathbf{u}_N^{(N)}] \in \mathbb{R}^{N \times N}, \\ \underline{V}_N &= [\mathbf{v}_1^{(N)}, \mathbf{v}_2^{(N)}, \dots, \mathbf{v}_N^{(N)}] \in \mathbb{R}^{N \times N}, \end{aligned}$$

with \underline{U}_N and \underline{V}_N being orthogonal. The singular values of \underline{A}_N are the scalars $s_j^{(N)}$, the vectors $\{\mathbf{u}_j^{(N)}\}$ and $\{\mathbf{v}_j^{(N)}\}$ are the left and right singular vectors of \underline{A}_N . The matrix norm of $\underline{A}_N = (a_{ij}^{(N)})$ that corresponds to the Hilbert-Schmidt norm of the operator A is the Frobenius norm $\|\underline{A}_N\|_F$ (see, for example [37]), which is defined by

$$\|\underline{A}_N\|_F^2 = \sum_{i=1}^N \sum_{j=1}^N \left(a_{ij}^{(N)} \right)^2 = \sum_{j=1}^N (s_j^{(N)})^2.$$

We will summarize some relationships between the SVE for operators in the infinite dimensional space $L^2(0, 1)$ and the SVD for real square matrices in the N -dimensional space in Table 1. This table is taken from a paper by Hansen [15]. Here, δ_{ij} is the Kronecker symbol, $\langle \cdot, \cdot \rangle_{L^2(0,1)}$ denotes the inner product in $L^2(0, 1)$, whereas $\langle \cdot, \cdot \rangle_2$ is the Euclidean inner product of vectors.

SVE	SVD
$\sigma_1 \geq \sigma_2 \geq \sigma_3 \dots \geq \sigma_n \geq \dots \rightarrow 0 \quad \text{as } n \rightarrow \infty$	$s_1^{(N)} \geq s_2^{(N)} \geq \dots s_n^{(N)} \geq 0$
$\ A\ _{HS}^2 = \sum_{j=1}^{\infty} \sigma_j^2$	$\ \underline{A}_N\ _F^2 = \sum_{j=1}^N \left(s_j^{(N)} \right)^2$
$\begin{aligned} \langle u_i, u_j \rangle_{L^2(0,1)} &= \delta_{ij} \\ \langle v_i, v_j \rangle_{L^2(0,1)} &= \delta_{ij} \end{aligned} \quad (i, j = 1, 2, \dots)$	$\begin{aligned} \langle \mathbf{u}_i^{(N)}, \mathbf{u}_j^{(N)} \rangle_2 &= \delta_{ij} \\ \langle \mathbf{v}_i^{(N)}, \mathbf{v}_j^{(N)} \rangle_2 &= \delta_{ij} \end{aligned} \quad (i, j = 1, 2, \dots, N)$
$\begin{aligned} \int_0^1 k(s, t) v_j(s) ds &= \sigma_j u_j(t) \\ \int_0^1 k(s, t) u_j(t) dt &= \sigma_j v_j(s). \end{aligned} \quad (j = 1, 2, \dots)$	$\begin{aligned} \underline{A}_N \mathbf{v}_j^{(N)} &= s_j^{(N)} \mathbf{u}_j^{(N)} \\ (\underline{A}_N)^T \mathbf{u}_j^{(N)} &= s_j^{(N)} \mathbf{v}_j^{(N)} \end{aligned} \quad (j = 1, 2, \dots, N)$

Table 1: Some properties of the SVE and the SVD

Unfortunately, finding the SVE is hard and as discussed in Section 2 even the asymptotics of singular values can only be determined analytically in special cases. Therefore, we exploit the SVD of matrices \underline{A}_N obtained as a Galerkin approximation (see, e.g., [3], [39] and [14]) of A

from (22). For relationships between SVE and SVD we refer to [10], [15] and [16]. Two papers considering mainly the condition numbers of associated Galerkin matrices are [1] and [39]. We use the Galerkin method in form of Hansen's moment method (see [15]) with orthonormal basis functions. It is a universal expansion method for discretizing a linear Fredholm integral equation of the first kind.

Let $\{\Psi_j\}$ and $\{\Phi_j\}$ be two sets of orthonormal basis functions in $L^2(0, 1)$, the left basis functions Ψ_1, \dots, Ψ_N in the interval $I_s = [0, 1]$ and the right basis functions Φ_1, \dots, Φ_N in the interval $I_t = [0, 1]$. Then we approximate the solution x of the integral equation (22) by

$$x(t) = \sum_{j=1}^N x_j \Phi_j(t).$$

Such a solution of (22) corresponds to a right-hand side

$$y(s) = \sum_{j=1}^N x_j \int_0^1 k(s, t) \Phi_j(t) dt \quad (0 \leq s \leq 1).$$

The inner product of (3.2) with the function Ψ_i in $L^2(0, 1)$ yields

$$\sum_{j=1}^N x_j \int_0^1 \int_0^1 k(s, t) \Phi_j(t) \Psi_i(s) dt ds = \int_0^1 y(s) \Psi_i(s) ds,$$

and hence the linear algebraic system

$$\sum_{j=1}^N a_{ij}^{(N)} x_j = y_i \quad (i = 1, 2, \dots, N),$$

$y_i = \langle y, \Psi_i \rangle_{L^2(0,1)}$ and $a_{ij}^{(N)} = \langle A \Phi_j, \Psi_i \rangle_{L^2(0,1)}$, which can be written in an easier notation as

$$\underline{A}_N \mathbf{x} = \mathbf{y}$$

with $\underline{A}_N = (a_{ij}^{(N)}) \in \mathbb{R}^{N \times N}$, $\mathbf{x} = (x_1, \dots, x_N)^T \in \mathbb{R}^N$ and $\mathbf{y} = (y_1, \dots, y_N)^T \in \mathbb{R}^N$. The system (3.2) is assumed to be an appropriate discretization of the integral equation (22).

In order to evaluate the decay rate of singular values of the operator A , we compute the singular values $s_i^{(N)}$ ($i = 1, \dots, N$) of the matrix \underline{A}_N assuming that they are sufficiently good approximations to the first N true singular values σ_i ($i = 1, \dots, N$) of the operator A .

Algorithm 3.2 (Galerkin method)

1. Choose a discretization level N and two sets of orthonormal basis functions $\{\Psi_j\}$ and $\{\Phi_j\}$ ($j = 1, \dots, N$) all defined on the interval $I_s = I_t = [0, 1]$.
2. Determine the matrix $\underline{A}_N = (a_{ij}^{(N)}) \in \mathbb{R}^{N \times N}$ with $a_{ij}^{(N)} = \langle A \Phi_j, \Psi_i \rangle_{L^2(0,1)}$ ($i, j = 1, \dots, N$).
3. Compute the singular values $s_1^{(N)} \geq s_2^{(N)} \geq \dots \geq s_N^{(N)}$ of the matrix \underline{A}_N .

The following proposition presents some assertions on the accuracy of approximations provided by Algorithm 3.2 based on [15] and [16].

Proposition 3.3 *Let the matrix \underline{A}_N denote as in Algorithm 3.2 and moreover let \tilde{A}_N be an integral operator acting in $L^2(0,1)$ with the degenerate kernel $\tilde{k}_N(s,t)$ given by*

$$\tilde{k}_N(s,t) = \sum_{i=1}^N \sum_{j=1}^N a_{ij}^{(N)} \Psi_i(s) \Phi_j(t)$$

that approximates the Hilbert-Schmidt operator A from (22). Then

$$\Delta_N^2 = \|A - \tilde{A}_N\|_{HS}^2 = \|A\|_{HS}^2 - \|\underline{A}_N\|_F^2$$

is valid and for the ordered singular values $s_i^{(N)}$ of the matrix \underline{A}_N and σ_i of the operator A the estimates

$$0 \leq \sigma_i - s_i^{(N)} \leq \Delta_N \quad (i = 1, \dots, N)$$

hold where a growing number N of basis functions improves the approximation of σ_i by $s_i^{(N)}$ in the sense of the inequalities

$$s_i^{(N)} \leq s_i^{(N+1)} \leq \sigma_i \quad (i = 1, \dots, N).$$

The sum of squares of the errors of the approximate singular values $s_i^{(N)}$ is bounded by

$$\sum_{i=1}^N [\sigma_i - s_i^{(N)}]^2 \leq \Delta_N^2.$$

Moreover, the true singular values σ_i of A are bounded in terms of the computed singular values $s_i^{(N)}$ of \underline{A}_N by

$$s_i^{(N)} \leq \sigma_i \leq [(s_i^{(N)})^2 + \Delta_N^2]^{\frac{1}{2}} \quad (i = 1, \dots, N).$$

For a proof we refer to [15]. This proposition implies that if $\Delta_N \rightarrow 0$ as $N \rightarrow \infty$, then the approximate singular values $s_i^{(N)}$ converge uniformly for all i to the true singular values σ_i . Furthermore, the Galerkin method always gives lower bounds for the first N eigenvalues of the operator A .

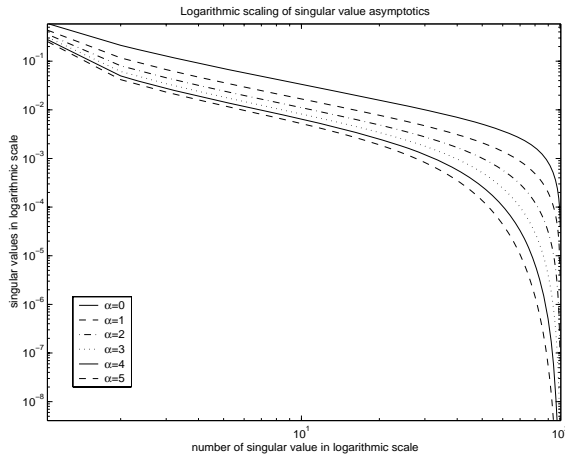


Figure 3: Computed singular values of \underline{A}_N with Galerkin method for $N = 100$, multiplier function $m(s) = s^\alpha$ and different values for α in logarithmic scales

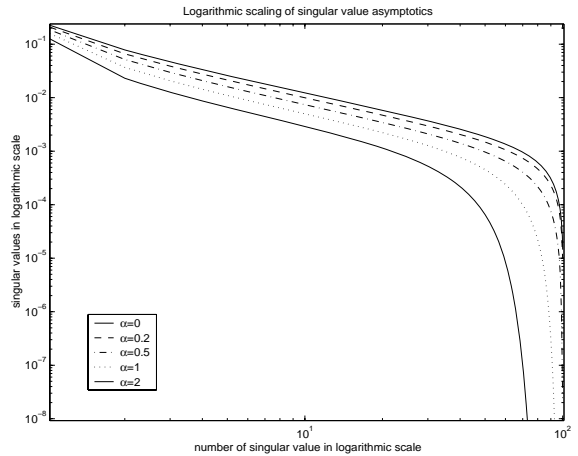


Figure 4: Computed singular values of \underline{A}_N with Galerkin method for $N = 100$, multiplier function $m(s) = e^{-\frac{1}{s^\alpha}}$ and different values for α in logarithmic scales

We tested the Galerkin method described above for the problem (22) with kernel k from (13) and the multiplier functions (5) and (6) for several values of α . The orthonormal basis functions $\{\Psi_j\}$ and $\{\Phi_j\}$ are simply chosen to give piecewise constant approximations to the singular functions. The interval $I_s = I_t = [0, 1]$ is divided into N subintervals $[s_i, s_{i+1}]$ and $[t_j, t_{j+1}]$ of length $h = 1/N$. The basis functions are then given by

$$\begin{aligned}\Phi_j(t) &= \begin{cases} \sqrt{t_j - t_{j-1}} & (t \in [t_{j-1}, t_j]) \\ 0 & \text{else} \end{cases} \quad \text{and} \\ \Psi_i(s) &= \begin{cases} \sqrt{s_i - s_{i-1}} & (s \in [s_{i-1}, s_i]) \\ 0 & \text{else} \end{cases}\end{aligned}$$

for all $i, j = 1, \dots, N$. Note that the matrix \underline{A}_N becomes a lower triangular matrix, due to the fact that the kernel $k(s, t)$ is only defined on the lower triangle.

In Figures 3 and 4 we observe nearly linear graphs of the singular values $s_i^{(N)}$ in logarithmic scales for all α and for i not too close to N . Detailed error analysis is carried out in [11]. The figures suggest that the singular value asymptotics in both cases is given by an expression $s_i^{(N)} \approx g(\alpha)/i$, where g are different monotonically decreasing functions depending on α for power type and exponential type multiplier functions. A detailed inspection of the functions g for different values of α again confirms the asymptotics (21) in the power case. For exponential type functions we got results that support conjecture (18).

3.3 A Rayleigh-Ritz method for symmetric kernels and the generalized eigenvalue problem

Finally we are going to consider the Rayleigh-Ritz method as a special case of Galerkin's method for symmetric kernels. We consider the positive definite, self-adjoint integral operator $C = A^*A$ with kernel K defined by (15) and in particular by (16), where we assume $m \in L^2(0, 1)$. Then solving the eigenvalue problem

$$C u = \lambda u \tag{23}$$

provides a nonincreasing sequence of eigenvalues λ_i which are squares of the singular values σ_i of the operator A under consideration.

We are going to use the same procedure as for the Galerkin method, but here we call it Rayleigh-Ritz procedure as in [7] and [31], since the Rayleigh quotient plays an important role. The Rayleigh-Ritz method was also investigated by Baker [3] who states that in the case of symmetric kernels $K(t, \tau)$ the Galerkin equations from the previous subsection are precisely the equations obtained from the Rayleigh-Ritz method.

Let $\{\Phi_j\} \subset L^2(0, 1)$ be a linearly independent set of basis functions in the interval $I = [0, 1]$. Notice that this time we need only one set of basis functions, since for an eigenvalue expansion we have just one set of eigenfunctions, whereas for a singular value expansion we have left and right singular functions (see the previous subsection). The set $\{\Phi_j\}$ also does not necessary have to be orthonormal. Then we approximate the eigenfunction u of operator equation (23) by

$$u(t) = \sum_{j=1}^N u_j \Phi_j(t).$$

Hence, equation (23) becomes

$$\sum_{j=1}^N u_j \int_0^1 K(t, \tau) \Phi_j(t) dt = l^{(N)} \sum_{j=1}^N u_j \Phi_j(\tau) \quad (0 \leq t \leq 1), \tag{24}$$

where $l^{(N)} \approx \lambda$ are the approximate eigenvalues if we choose an N -dimensional subspace. Multiplying (24) by $\Phi_i(t)$ and integrating with respect to t yields

$$\sum_{j=1}^N u_j \int_0^1 \int_0^1 K(t, \tau) \Phi_j(t) \Phi_i(\tau) dt d\tau = l^{(N)} \sum_{j=1}^N u_j \int_0^1 \Phi_j(\tau) \Phi_i(\tau) d\tau,$$

or, in an easier notation

$$\sum_{j=1}^N c_{ij}^{(N)} u_j = l^{(N)} \sum_{j=1}^N d_{ij}^{(N)} u_j,$$

where we have $c_{ij}^{(N)} = \langle C \Phi_j, \Phi_i \rangle_{L^2(0,1)}$ and $d_{ij}^{(N)} = \langle \Phi_j, \Phi_i \rangle_{L^2(0,1)}$ for all $i, j = 1, \dots, N$. Therefore the eigenvalue equation (23) is approximated by the generalized matrix eigenvalue problem

$$\underline{C}_N \mathbf{u} = l^{(N)} \underline{D}_N \mathbf{u}, \quad (25)$$

where $\underline{C}_N = (c_{ij}^{(N)}) \in \mathbb{R}^{N \times N}$, $\underline{D}_N = (d_{ij}^{(N)}) \in \mathbb{R}^{N \times N}$ and $\mathbf{u} \in \mathbb{R}^N$. Notice that for orthonormal sets of basis functions $\{\Phi_j\}$ we have $d_{ij}^{(N)} = \delta_{ij}$ and therefore \underline{D}_N is a unit matrix. Then (25) becomes a standard matrix eigenvalue problem.

By solving the generalized matrix eigenproblem (25) we will get approximations to the true eigenvalues of $C = A^*A$. This was shown in Cochran [7] and is actually a special case of the results obtained by Hansen [15]. First we will summarize the algorithm.

Algorithm 3.4 (Rayleigh-Ritz method)

1. Choose a discretization level N and a set of linearly independent basis functions $\{\Phi_j \in L^2(0,1)\}$ ($j = 1, \dots, N$) defined on the interval $I = [0, 1]$.
2. Determine the matrices $\underline{C}_N = (c_{ij}^{(N)}) \in \mathbb{R}^{N \times N}$ and $\underline{D}_N = (d_{ij}^{(N)}) \in \mathbb{R}^{N \times N}$ with $c_{ij}^{(N)} = \langle C \Phi_j, \Phi_i \rangle_{L^2(0,1)}$ and $d_{ij}^{(N)} = \langle \Phi_j, \Phi_i \rangle_{L^2(0,1)}$ for $i, j = 1, \dots, N$.
3. Compute the eigenvalues $l_1^{(N)} \geq l_2^{(N)} \geq \dots \geq l_N^{(N)} > 0$ of the generalized eigenvalue problem $\underline{C}_N \mathbf{u} = l^{(N)} \underline{D}_N \mathbf{u}$.

The approximation properties of this algorithm with respect to the eigenvalues of the operator C are given by the following proposition which is due to [3] (for a proof see [3, p.316]).

Proposition 3.5 *Let the matrices \underline{C}_N and \underline{D}_N denote as in Algorithm 3.4. Then there exist positive algebraic eigenvalues $l_1^{(N)} \geq l_2^{(N)} \geq \dots \geq l_N^{(N)}$ of the generalized matrix eigenvalue problem (25) which are increasingly better approximations to the true N largest positive eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ of the operator C , i.e., for an increasing number N of basis functions the approximation is improved in the sense of the estimates*

$$l_i^{(N)} \leq l_i^{(N+1)} \leq \lambda_i \quad (i = 1, \dots, N). \quad (26)$$

We are going to apply the Rayleigh-Ritz method to the integral operator $C = A^*A$ with the symmetric kernel K from (16) in order to compute its eigenvalues approximately. We again

consider multiplier function (5) and (6) for several values of α . The basis functions $\{\Phi_j\}$ are chosen to be piecewise linear interpolations, i.e.,

$$\Phi_j(t) = \begin{cases} \frac{t - t_{j-1}}{h} & (t \in [t_{j-1}, t_j]) \\ \frac{t_{j+1} - t}{h} & (t \in [t_j, t_{j+1}]) \\ 0 & \text{else} \end{cases} \quad (j = 1, \dots, N)$$

with equidistant subintervals of length $h = 1/N + 1$. For computational details of the use of the algorithm 3.4 we again refer to [11].

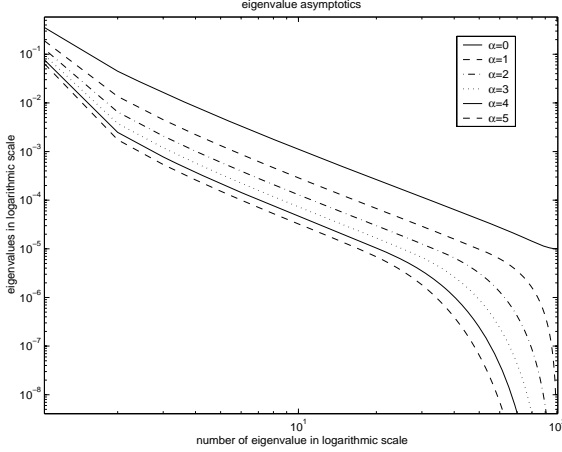


Figure 5: Approximated eigenvalues of A^*A with Rayleigh-Ritz method for $N = 100$, multiplier function $m(s) = s^\alpha$ and different values for α in logarithmic scales

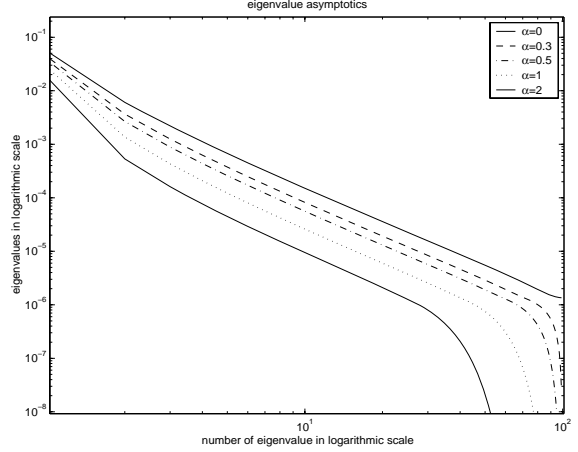


Figure 6: Approximated eigenvalues of A^*A with Rayleigh-Ritz method for $N = 100$, multiplier function $m(s) = e^{-\frac{1}{s^\alpha}}$ and different values for α in logarithmic scales

The behavior of eigenvalues $l_i^{(N)}$ obtained by Algorithm 3.4 for growing i and multiplier functions (5) and (6) is shown in Figures 5 and 6. We observe that in the mid part the graphs the lines are nearly parallel in logarithmic scaling with a negative vertical shift for growing values of α in both cases. For large i , i.e. $i \approx N$, the confidence in the computed eigenvalues gets lost as a consequence of the ill-conditioning of the matrix \underline{C}_N . A detailed inspection of the slope of all graphs implies $l_i^{(N)} \approx h(\alpha)/i^2$ with a monotonically decreasing positive function $h(\alpha)$ and also confirms the asymptotics (21) in the power case taking into account that the singular values of A are given by the square roots of the positive eigenvalues of A^*A . Tables of computed eigenvalues based on the Rayleigh-Ritz method are presented in [11]. In the exponential case the tables are satisfactorily compatible with the conjecture (18).

Note that all three independent numerical approaches confirm the fact that the degree $\mu = 1$ of ill-posedness for linear integral equations (22) with kernel (13) remains true if the multiplier functions m have single zeros of arbitrary order.

3.4 Some further investigations on multiplier functions without zeros

In this subsection we complete the numerical studies of Section 3 by considering small perturbations $\delta > 0$ of a multiplier function m with a single zero such that the perturbed function \tilde{m}

has no zeros. In particular, we are going to consider three different functions $\tilde{m}(s)$ given by

$$\begin{aligned}\tilde{m}_1(s) &= m(s) + \delta & (0 \leq s \leq 1), \\ \tilde{m}_2(s) &= m(s + \delta) & (0 \leq s \leq 1), \\ \tilde{m}_3(s) &= \begin{cases} m(s) & (\delta \leq s \leq 1) \\ m(\delta) & (0 < s < \delta). \end{cases}\end{aligned}$$

Figure 7 shows the three types of functions for the special case $m(s) = s^2$ and $\delta = 0.1$. Notice that $\tilde{m}(s)$ tends to $m(s)$ uniformly for all s if $\delta \rightarrow 0$.

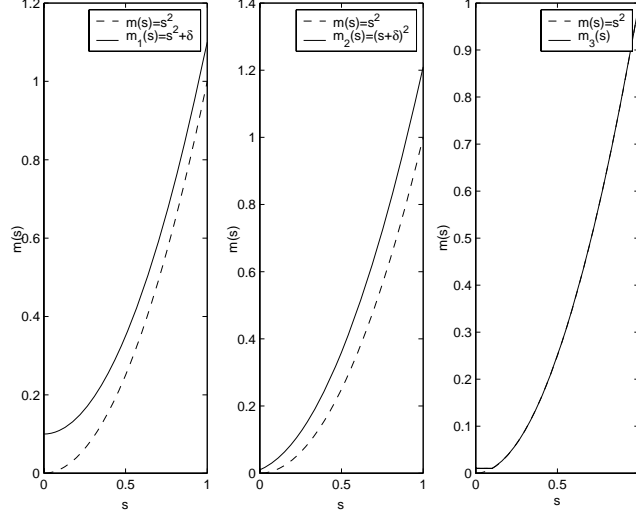


Figure 7: Perturbed multiplier functions $\tilde{m}(s)$ for $m(s) = s^2$ and $\delta = 0.1$

Before stating any numerical outcomes, we want to mention some inequalities obtained analytically. Namely, for positive lower and upper bounds c and C with

$$0 < c \leq \tilde{m}(s) \leq C \quad (0 \leq s \leq 1)$$

we have as an immediate consequence of Lemma 2.1

$$\frac{2c}{\pi(2n-1)} \leq \sigma_n(A) \leq \frac{2C}{\pi(2n-1)}.$$

Obviously we may compute c and C for the perturbed multiplier functions $\tilde{m}(s)$. For multiplier function $m(s) = s^\alpha$ we then get with J from (3)

$$\delta \sigma_n(J) \leq \sigma_n(A) \leq (1 + \delta) \sigma_n(J),$$

for $\tilde{m}_1(s)$,

$$\delta^\alpha \sigma_n(J) \leq \sigma_n(A) \leq (1 + \delta)^\alpha \sigma_n(J),$$

for $\tilde{m}_2(s)$ and

$$\delta^\alpha \sigma_n(J) \leq \sigma_n(A) \leq \sigma_n(J),$$

for $\tilde{m}_3(s)$. We may do the same calculation for the perturbed multiplier function $\tilde{m}(s)$ with $m(s) = e^{-\frac{1}{s^\alpha}}$ and obtain

$$\delta \sigma_n(J) \leq \sigma_n(A) \leq (e^{-1} + \delta) \sigma_n(J),$$

for $\tilde{m}_1(s)$,

$$e^{-\frac{1}{\delta^\alpha}} \sigma_n(J) \leq \sigma_n(A) \leq e^{-\frac{1}{(1+\delta)^\alpha}} \sigma_n(J),$$

for $\tilde{m}_2(s)$ and

$$e^{-\frac{1}{\delta^\alpha}} \sigma_n(J) \leq \sigma_n(A) \leq e^{-1} \sigma_n(J),$$

for $\tilde{m}_3(s)$. In all three cases for the two types of multiplier functions the degree μ of ill-posedness of operator A is one.

We are now going to calculate the singular values of the operator A with multiplier functions \tilde{m} approximately for several perturbations δ , by using the Galerkin method. Just for a better overview we only consider the special multiplier functions $m(s) = s^2$ and $m(s) = e^{-\frac{1}{s}}$. In the left part of Table 2 we summarize the quotient

$$Q = \frac{\sigma_n^{approx}(A)}{\sigma_n(J)},$$

for $m(s) = s^2$ and several values of δ in order to compare these results to our previous ones.

δ	$\tilde{m}_1(s)$	$\tilde{m}_2(s)$	$\tilde{m}_3(s)$
1	1.333	2.314	1
0.5	0.833	1.072	0.414
0.1	0.434	0.440	0.332
0.01	0.341	0.341	0.332
0.001	0.333	0.333	0.332

δ	$\tilde{m}_1(s)$	$\tilde{m}_2(s)$	$\tilde{m}_3(s)$
1	1.333	2.333	1
0.5	0.833	1.083	0.348
0.1	0.433	0.443	0.333
0.01	0.343	0.343	0.333
0.001	0.334	0.334	0.333

Table 2: Left table: The quotient $Q = \sigma_n^{approx}(A)/\sigma_n(J)$ for $\tilde{m}_i(s)$ ($i = 1, 2, 3$), $m(s) = s^2$, $N = 1000$ and several values of δ . Right table: The integral $\int_0^1 \tilde{m}_i(s)ds$ for $\tilde{m}_i(s)$ ($i = 1, 2, 3$), $m(s) = s^2$ and several values of δ

Firstly we observe that for $\delta \rightarrow 0$ the singular value asymptotics and the coefficient Q is the same as the one obtained in the previous section, i.e. (here with $\alpha = 2$)

$$Q = \frac{1}{\alpha + 1}.$$

Since we observe a certain relation between Q and the integral $\int_0^1 \tilde{m}_i(s)ds$ we are going to determine this integral (which can be done analytically in this case) and summarize the values in the right part of Table 2. >From those two tables we may conjecture that

$$\sigma_n^{approx}(A) \sim \int_0^1 m(s)ds \cdot \sigma_n(J) = \int_0^1 m(s)ds \cdot \frac{2}{\pi(2n-1)}, \quad (27)$$

for the integral operator $A = M \circ J$.

Now we consider the multiplier function $m(s) = e^{-\frac{1}{s}}$. In the Table 3 we summarize the quotient Q in the left part and the integral $\int_0^1 \tilde{m}_i(s)ds$ in the right part of this table. We observe the same relationship between Q and the integral $\int_0^1 \tilde{m}_i(s)ds$ as formulated in equation (27).

A proof for this relationship (27) for the case of power type functions has been found, see Proposition 2.2. For the case of exponential type functions the analytical considerations could not provide assertions on the degree of ill-posedness, but the numerical results are rather satisfactory.

δ	$\tilde{m}_1(s)$	$\tilde{m}_2(s)$	$\tilde{m}_3(s)$
1	1.144	0.502	0.367
0.5	0.645	0.350	0.196
0.1	0.244	0.186	0.148
0.01	0.156	0.152	0.148
0.001	0.149	0.148	0.148

δ	$\tilde{m}_1(s)$	$\tilde{m}_2(s)$	$\tilde{m}_3(s)$
1	1.148	0.504	0.368
0.5	0.648	0.353	0.197
0.1	0.248	0.187	0.148
0.01	0.158	0.152	0.148
0.001	0.149	0.149	0.148

Table 3: Left table: The quotient $Q = \sigma_n^{approx}(A)/\sigma_n(J)$ for $\tilde{m}_i(s)$ ($i = 1, 2, 3$), $m(s) = e^{-\frac{1}{s}}$, $N = 200$ and several values of δ . Right table: The integral $\int_0^1 \tilde{m}_i(s)ds$ for $\tilde{m}_i(s)$ ($i = 1, 2, 3$), $m(s) = e^{-\frac{1}{s}}$ and several values of δ

4 On Tikhonov regularization for the linear equation

In this paragraph we consider the Tikhonov regularization for the linear operator equation (1). We again focus on composite linear operators $A = M \circ J$ defined by formula (2) mapping in $X = Y = L^2(0, 1)$ and study here the influence of varying multiplier functions m possessing essential zeros on the error and convergence of regularized solutions.

The method of Tikhonov regularization is standard method for the stable approximate solution of ill-posed equations (1) (see, e.g., [35] and [4], [5], [8], [13], [18], [30]), where regularized solutions x_β depending on a regularization parameter $\beta > 0$ are obtained by solving the extremal problem

$$\|Ax - y\|^2 + \beta \|x\|^2 \longrightarrow \min, \quad \text{subject to } x \in X$$

with uniquely determined minimizer

$$x_\beta = (A^*A + \beta I)^{-1}A^*y$$

It is well-known that for an exact right-hand side $y = Ax_0$ we obtain convergence rates

$$\|x_\beta - x_0\| = \mathcal{O}(\sqrt{\beta}) \quad \text{as } \beta \rightarrow 0$$

of regularized solutions x_β to the exact solution x_0 if x_0 satisfies a source condition

$$x_0 = A^*v \quad (v \in Y). \quad (28)$$

In general, a growing degree of ill-posedness of (1) corresponds to a growing strength of the condition (28) imposed on the solution element x_0 .

Now we compare the strength of condition (28) for the case $A = J$ with the simple integration operator J defined by formula (3) written as

$$x_0(t) = [J^*v](t) = \int_t^1 v(s)ds \quad (0 \leq t \leq 1; v \in L^2(0, 1), \|v\| \leq R_0) \quad (29)$$

and the strength of condition (28) for the case $A = M \circ J$ with the composite integral operator from (2) with weights m having zeros. Provided that weight functions m occur we can write (28) as

$$x_0(t) = [J^*M^*v](s) = [J^*Mv](t) = \int_t^1 m(s)v(s)ds \quad (0 \leq t \leq 1; v \in L^2(0, 1)). \quad (30)$$

If we assume that the multiplier function m has an essential zero only at $t = 0$, then the condition (29) that can be reformulated as

$$x'_0 \in L^2(0, 1) \quad (\text{i.e. } x_0 \in H^1(0, 1)) \quad \text{with} \quad x_0(1) = 0$$

is obviously weaker than the condition (30) which is equivalent to

$$\frac{x'_0}{m} \in L^2(0, 1) \quad \text{with} \quad x_0(1) = 0, \quad (31)$$

since the new factor $\frac{1}{m}$ occurring in (31) is not in $L^\infty(0, 1)$. Consequently in order to satisfy the source condition (31), the generalized derivative of the function x_0 has to compensate in some sense the pole of $\frac{1}{m}$ at $t = 0$. The requirement of compensation grows when the decay rate of $m(t) \rightarrow 0$ as $t \rightarrow 0$ grows. Hence the strength of the requirement (31) imposed on x_0 grows for the families (5) and (6) of multiplier functions m when the exponents r and ρ increase. Moreover, for the exponential type functions (6) the condition (31) is stronger than for the power type functions (5).

Let us assume that the canonical source condition (29) is satisfied for the simple integration operator J defined by formula (3), but we do not assume the stronger condition (30). In order to investigate this situation we use a lemma formulated and proved in Baumeister's book [5, Theorem 6.8] (see also [26]):

Lemma 4.1 *If we introduce the distance function*

$$d(R) = \inf \{ \|x_0 - A^* v\| : v \in Y, \|v\| \leq R \},$$

then we have

$$\|x_\beta - x_0\| \leq \sqrt{(d(R))^2 + \beta R^2} \leq d(R) + \sqrt{\beta} R \quad (32)$$

for all $\beta > 0$ and $R > 0$.

Under the assumption (29) Lemma 4.1 applies for $A = M \circ J$ and helps to evaluate the error $\|x_\beta - x_0\|$ as a function of $\beta > 0$ if we additionally consider the estimate

$$d(R_0) \leq \|J^* v - A^* v\| = \|J^*(v - Mv)\| \quad \text{with} \quad \|v\| \leq R_0.$$

Namely, then we have

$$(d(R))^2 \leq \int_0^1 \left(\int_s^1 (1 - m(t)) v(t) dt \right)^2 ds \leq \left(\int_0^1 (1 - m(t))^2 dt \right) \|v\|^2$$

and from (32) we derive the error estimate

$$\|x_\beta - x_0\| \leq R_0 \sqrt{\beta + \frac{(d(R_0))^2}{R_0^2}} \leq R_0 \sqrt{\beta + \int_0^1 (1 - m(t))^2 dt}. \quad (33)$$

If we compare the right term in formula (33) with the function $R_0 \sqrt{\beta}$ that would occur as an error bound whenever (30) would hold, we can see that the former function is obtained by the latter by applying a shift to the left with value $\int_0^1 (1 - m(t))^2 dt \geq 0$. A positive shift destroys the

convergence rate. However if the shift is small and $\beta > 0$ not too small, then the regularization error is nearly the same as in the case (30).

If, for example, values of the continuous non-decreasing multiplier function m as plotted in Figure 8 deviate from one only on a small interval $[0, \varepsilon)$, i.e.,

$$\begin{aligned} m(t) &= 1 & (\varepsilon \leq t \leq 1), \\ m(t) &\rightarrow 0 & \text{as } t \rightarrow 0, \end{aligned}$$

we have

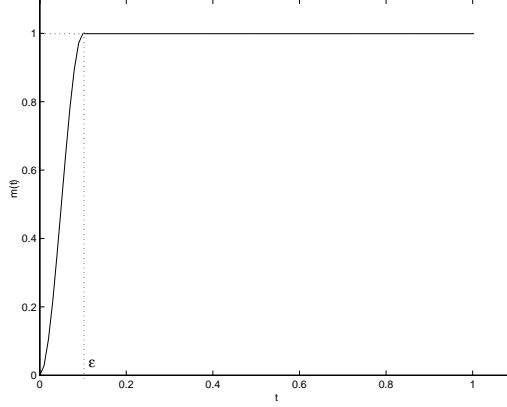


Figure 8: Function $m(t)$ deviating from 1 only on a small interval $0 \leq t \leq \varepsilon$.

$$\|x_\beta - x_0\| \leq R_0 \sqrt{\beta + \varepsilon} \quad (34)$$

and the influence of the multiplier function disappears as ε tends to zero. As the consideration above and formula (34) show, the decay rate of $m(t) \rightarrow 0$ as $t \rightarrow 0$ as a power (cf. (5)) or exponentially (cf. (6)) in a neighbourhood of $t = 0$ is in that case without meaning for the regularization error. Only the integrals $\int_0^1 (1 - m(t))^2 dt$ play an important role. This is analogous to the character of the integral $\int_0^1 m(t) dt$ which is an essential factor for the singular values of $A = M \circ J$ (see Proposition 2.2 and Conjecture 2.3 above).

5 On Tikhonov regularization for the nonlinear equation

Finally, we are going to treat the nonlinear inverse problem (7) focused on the special case (8) immediately by using Tikhonov regularization along the lines of the seminal paper [9] of Engl, Kunisch and Neubauer (see also the book [8]). Regularized solutions $x_\beta^\delta \in D(F)$ are stable approximate solutions of (7) based on noisy data $y^\delta \in Y$, where $y = F(x_0)$ with $x_0 \in D(F)$ represents the exact right-hand side and $\delta > 0$ with $\|y - y^\delta\| \leq \delta$ is the noise level. In this context, the elements x_β^δ are minimizers of the extremal problem

$$\|F(x) - y^\delta\|^2 + \beta \|x - x^*\|^2 \rightarrow \min, \quad \text{subject to } x \in D(F),$$

where $x^* \in X$ is an initial guess for the solution x_0 to be determined. If F is continuous and weakly closed (for a discussion of those properties for the special case (8) see [21]), then for all $\beta > 0$ regularized solutions x_β^δ exist (not necessarily unique) and depend stably on the data y^δ . Moreover, the theory of [9] on convergence and convergence rates applies. We consider here the source condition

$$x_0 - x^* = F'(x_0)^* v \quad (v \in Y), \quad (35)$$

but we assume that (35) is satisfied only in an approximate manner and the convergence rate

$$\|x_{\beta(\delta)}^\delta - x_0\| = \mathcal{O}(\sqrt{\delta}) \quad \text{as} \quad \delta \rightarrow 0$$

proven for the a priori parameter choice $\beta(\delta) \sim \delta$ in [9] cannot be expected. We present a proposition, which is a version of Theorem 4.1 in [29] proven by Lukaschewitsch in [28, theorem 2.2.3]:

Proposition 5.1 *Let $F : D(F) \subset X \rightarrow Y$ be a continuous nonlinear operator mapping between the Hilbert spaces X and Y and let the following assumptions hold:*

- (i) *The domain $D(F)$ is convex.*
- (ii) *The operator F is weakly (sequentially) closed.*
- (iii) *The element $x_0 \in D(F)$ is an x^* -minimum-norm solution, i.e.,*

$$\|x_0 - x^*\| = \min\{\|x - x^*\| : F(x) = y, x \in D(F)\}.$$

- (iv) *For some radius $\rho > 0$ there is a ball $B_\rho(x_0)$ with centre x_0 such that the Fréchet derivative $F'(x)$ of F exists for all $x \in D(F) \cap B_\rho(x_0)$ and an estimate*

$$\|F(x) - F(x_0) - F'(x_0)(x - x_0)\| \leq \frac{L}{2} \|x - x_0\|^2 \quad (36)$$

is satisfied for some constant $L > 0$ and all $x \in D(F) \cap B_\rho(x_0)$.

- (v) *Let exist a number $\theta \geq 0$ such that there is an element $w \in Y$ with*

$$c = L\|w\| < 1$$

and

$$\|x_0 - x^* - F'(x_0)^*w\| \leq \theta.$$

Then, for an a priori choice $\beta = K\delta$ with some constant $K > 0$ and $0 < \delta \leq \delta_0$ and if $\rho > 2\|x_0 - x^\| + \sqrt{\frac{\delta_0}{K}}$, we have the estimate*

$$\|x_{\beta}^\delta - x_0\| \leq \frac{1}{\sqrt{1-c}} \left(\left(\frac{1}{\sqrt{K}} + \frac{\sqrt{K}c}{L} \right) \sqrt{\delta} + \sqrt{2\rho} \sqrt{\theta} \right).$$

Now we consider the specific class (8) of nonlinear problems and apply Proposition 5.1, where we assume that the generator function κ is smooth enough such that the nonlinear operator F is continuous, weakly closed and $F'(x_0)$ from (11) defines a bounded linear operator in $L^2(0, 1)$ which is a Lipschitz continuous Fréchet derivative of F satisfying the inequality (36) for some $L > 0$ (for details see again [21]). An example for estimating θ can be given if we assume a weaker source condition

$$(x_0 - x^*)(t) = \int_t^1 w(s)ds \quad (0 \leq t \leq 1)$$

for some $w \in L^2(0, 1)$ and $c = L\|w\| < 1$ instead of (35). Then we have with $F'(x_0)^* = J^* \circ M^* = J^* \circ M$

$$\|x_0 - x^* - F'(x_0)^*w\| = \|J^*w - J^*Mw\| \leq \theta = \left(\sqrt{\int_0^1 (1 - m(t))^2 dt} \right) \|w\|.$$

For a non-decreasing multiplier function $m(t)$ with zero at $t = 0$ as shown in Figure 8 deviating from one only on an interval $0 \leq t \leq \varepsilon$ we get

$$\|x_\beta^\delta - x_0\| \leq \frac{1}{\sqrt{1-c}} \left(\left(\frac{1}{\sqrt{K}} + \frac{\sqrt{K}c}{L} \right) \sqrt{\delta} + \sqrt{\frac{2\rho}{L}} \sqrt[4]{\varepsilon} \right)$$

and therefore a small influence on the convergence rate for small $\varepsilon > 0$. Again only an integral $\int_0^1 (1 - m(t))^2 ds$ and not the decay rate of $m(t) \rightarrow 0$ as $t \rightarrow 0$ influences the regularization properties. This is the same qualitative result as observed for the Tikhonov regularization applied to the linearized problem in the preceding section of this paper.

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